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On a Conjecture of Bondy

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Bondy a conjecturé que: si G est un graphe k-connexe, où $k \geqslant 2$, tel que la somme des degrés de tout stable à $k+1$ éléments est au moins m, alors G contient un cycle de longueur au moins: $\mathrm{Min}(2m/(k+1), n)$ ($n =$ ordre de G). Nous prouvons ici que ce résultat est vrai. © 1985 Academic Press, Inc.

Bondy conjectured [1] that: if G is a k-connected graph, where $k \geqslant 2$, such that the degree-sum of any $k+1$ independent vertices is at least m, then G contains a cycle of length at least: $\mathrm{Min}(2m/(k+1), n)$ (n denotes the order of G). We prove here that this result is true. © 1985 Academic Press, Inc.

INTRODUCTION AND NOTATION

Bondy has shown in [1]:

THEOREM (Bondy). *Let G be a 2-connected graph such that the degree-sum of any three independent vertices is at least m, with $m \geqslant n+2$ (where n denotes the order of G); then G contains a cycle of length at least* $\mathrm{Min}(n, 2m/3)$.

In [1 and 2], he conjectured that the same result is true without the condition $m \geqslant n+2$ and can be generalised to a k-connected graph ($k \geqslant 3$). We have proved that result.

THEOREM (Fournier and Fraisse). *Let G be a k-connected graph where $k \geqslant 2$, such that the degree-sum of any $k+1$ independent vertices is at least m. Then G contains a cycle of length at least* $\mathrm{Min}(n, 2m/(k+1))$.

We use the following notation:

- If H is a subgraph of G, $|H|$ is the number of vertices of H.
- Let C be a cycle of G; we suppose that we have given an orientation to C. If a and b are vertices of C, $[a, b]_C$ denotes the subgraph of C

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which is the path with ends a and b whose vertices are met on C when one goes on C from a to b. If a is a vertex of C, a^+ (resp. a^-) is the first vertex on C after (resp. before) a; a^{+l} (resp. a^{-l}) is the lth vertex on C after (resp. before) a. If S is a set of vertices of C, $S^+ = \{a^+ \mid a \in S\}$.

- If P is a path of G and if a and b are two vertices of P, $[a, b]_P$ denotes the subgraph of P which is the path with ends a and b.
- If H is a subgraph of G and if a is a vertex of G, $\Gamma_H(a)$ is the set of neighbours of a in H, and $d_H(a) = |\Gamma_H(a)|$.
- A C-path is a path in G such that only its ends are on C, of length at least 2.
- If a and b are two adjacent vertices of G, (a, b) denotes the edge with ends a and b.

Proof of the Theorem. Let C be a cycle of G, of maximum length. We assume that C is not hamiltonian. Let $R = G \backslash C$ and we give an arbitrary orientation to C.

LEMMA 1. *Let P be a path of maximal length among all the paths with a given end, a, on C and all the other vertices not on C (Fig. 1). Let x be the second end of P. Then there exists a C-path which contains x and all its neighbours in R, and such that one of its ends on C is a.*

Though this lemma is almost contained in Dirac's paper ([3, pp. 74, 75]), in order to be complete, we prove it here.

Proof of Lemma 1. If $d_R(x) = 0$ or 1, Lemma 1 immediately follows from the fact that G is 2-connected. Assume $d_R(x) \geqslant 2$. P being of maximum length, all the neighbours of x in R are on P. Let x' be the neighbour of x in R "the furthest" on P from x. (See Fig. 2.) Let C' be the cycle $[x, x']_P \cup (x', x)$. Let P_1 and P_2 be two disjoint paths between C and C' and let $Q = [x', a]_P$.

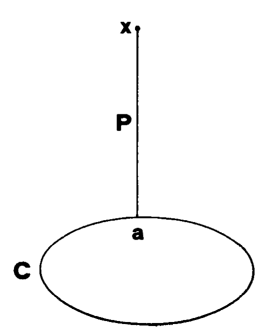

FIGURE 1

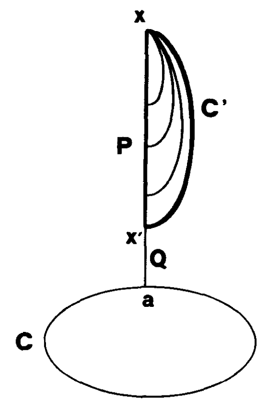

FIGURE 2

We shall construct two disjoint paths P_3 and P_4 between C and C' such that:

- x' is the end on C' of P_3
- a is the end on C of P_3 or P_4.

1st Case. If P_1 (resp. P_2) does not intersect Q, we set $P_3 = Q$, $P_4 = P_1$ (resp. P_2).

2nd Case. We suppose P_1 and P_2 intersect Q. Let r be the vertex nearest x' along Q and s the vertex nearest a along Q.

(a). r and s are on the same path, P_1, for example (Fig. 3):

$$P_3 = [x', r]_Q \cup [r, s]_{P_1} \cup [s, a]_Q$$
$$P_4 = P_2.$$

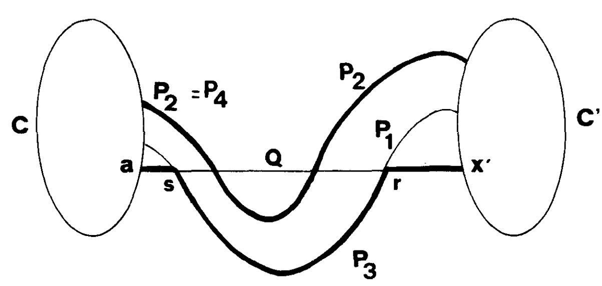

FIGURE 3

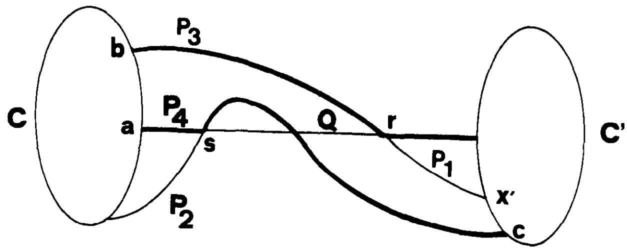

FIGURE 4

(b). $r \in P_1$, $s \in P_2$. Let b be the end of P_1 on C and c the end of P_2 on C'. We set (Fig. 4):

$$P_3 = [x', r]_Q \cup [r, b]_{P_1}$$

$$P_4 = [c, s]_{P_2} \cup [s, a]_Q.$$

In all the cases, we have obtained the paths P_3 and P_4.

Let y be the end of P_4 on C'. We orient C' to have $x \in [x', y]_{C'}$. We have two possibilities:

- If all the neighbours of x in P are on $[x', y]_{C'}$, we easily construct the C-path of Lemma 1 (Fig. 5).
- Suppose x has at least one neighbour on $[y^+, x'^-]_{C'}$. Let v be the neighbour of x on $[y^+, x'^-]_{C'}$ nearest y along C'. The C-path (Fig. 6)

$$P_3 \cup [v, x']_{C'} \cup (x, v) \cup [x, y]_{C'} \cup P_4$$

contains x and all its neighbours in R.

LEMMA 2. *Let us choose a, P, x as in Lemma* 1. *We have*

$$d_R(x) \leqslant \frac{|C|-4}{2}.$$

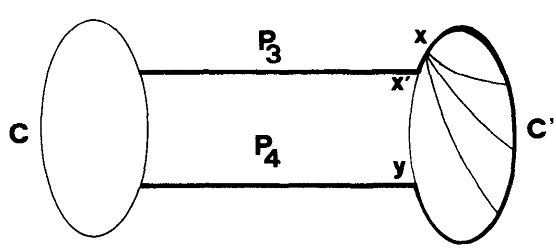

FIGURE 5

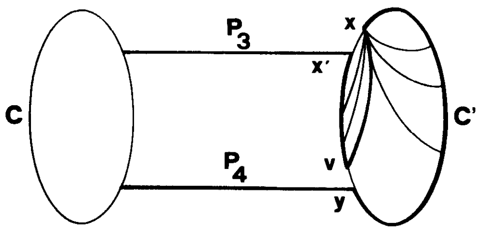

FIGURE 6

Proof of Lemma 2. With the C-path obtained in Lemma 1, we can construct (Fig. 7) a new cycle of length at least

$$\frac{|C|+2}{2}+d_R(x)+1.$$

Therefore $(|C|+2)/2+d_R(x)+1 \leqslant |C|$ which completes the proof of Lemma 2.

LEMMA 3. *We choose a, P, x as in Lemma* 1. *We have*

$$d(x) \leqslant |C|/2.$$

Proof of Lemma 3. C being of maximum length, x is not adjacent to the vertex a^{+l} if $1 \leqslant l \leqslant \mathrm{Min}(|P|-1, |C|-1)$.

We have:

- $|P|-1 \geqslant d_R(x)+1$ by the choice of P.
- $|C|-1 \geqslant 2d_R(x)+3$ from Lemma 2.

Therefore, x is not adjacent to a^{+l} for

$$1 \leqslant l \leqslant d_R(x)+1.$$

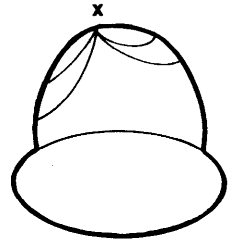

FIGURE 7

In the same way, x is not adjacent to a^{-l} for

$$1 \leqslant l \leqslant d_R(x) + 1.$$

Now, from Lemma 2,

$$2(d_R(x)+1) \leqslant 2\left(\frac{|C|-4}{2}+1\right)$$
$$\leqslant |C| - 2,$$

hence, the set of vertices $\{a^{+l} \mid 1 \leqslant l \leqslant d_R(x)+1\}$ does not intersect the set of vertices $\{a^{-l} \mid 1 \leqslant l \leqslant d_R(x)+1\}$.

Moreover, C being of maximum length, two consecutive vertices of C are never simultaneously adjacent to x:

$$d_C(x) \leqslant 1 + \frac{|C| - 1 - 2[d_R(x)+1] + 1}{2}$$

$$d_C(x) \leqslant \frac{|C|}{2} - d_R(x); \qquad d(x) \leqslant \frac{|C|}{2}.$$

LEMMA 4. *For a C-path, which is not reduced to an edge, of ends a_1 and a_2 on C:*

$$d_C(a_1^+) + d_C(a_2^+) \leqslant |C|.$$

Proof of Lemma 4. C being of maximum length:

$$\Gamma_C^+(a_1^+) \cap \Gamma_C(a_2^+) \cap [a_2^{++}, a_1] = \varnothing \qquad \text{(Fig. 8)},$$

so that

(1) $d_{[a_2^+, a_1^-]}(a_1^+) + d_{[a_2^{++} a_1]}(a_2^+) \leqslant |[a_2^{++}, a_1]|.$

Similarly

(2) $d_{[a_1^{++} a_2]}(a_1^+) + d_{[a_1^+ a_2^-]}(a_2^+) \leqslant |[a_1^{++}, a_2]|.$

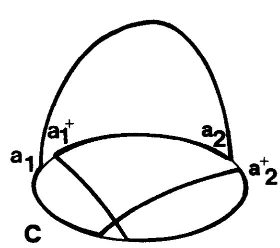

FIGURE 8

Summing (1) and (2) yields

$$d_C(a_1^+) + d_C(a_2^+) \leqslant |C|.$$

LEMMA 5. *We choose a, P, x as in Lemma* 1. *We have*

$$d(x) + d_C(a^+) \leqslant |C|.$$

Proof of Lemma 5.

First Case. $d_{C\setminus\{a\}}(x) = 0$. From Lemma 1, there exists a C-path, one of its ends being a, containing x and all its neighbours in R. Let b be the other end of this C-path. a^+ is not adjacent to b^{+l} if $1 \leqslant l \leqslant d_R(x) + 1$, otherwise C would not be of maximum length (Fig. 9),

so that

$$d_C(a^+) \leqslant |C| - (d_R(x) + 1)$$

$$d_C(a^+) + d_R(x) \leqslant |C| - 1$$

$$d_C(x) \leqslant 1$$

and

$$d_C(a^+) + d(x) \leqslant |C|.$$

Second Case. $d_{C\setminus\{a\}}(x) \geqslant 1$. Let b be the neighbour of x on $C\setminus\{a\}$ such that:

$$d_{[b^+, a^-]_C}(x) = 0.$$

Necessarily: $|[b^+, a^-]_C| \geqslant d_R(x) + 1$. Moreover, a^+ is not adjacent to any of the vertices b^{+l} where

$$1 \leqslant l \leqslant d_R(x) + 1 \qquad \text{(Fig. 10)}.$$

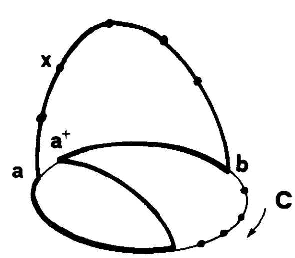

FIGURE 9

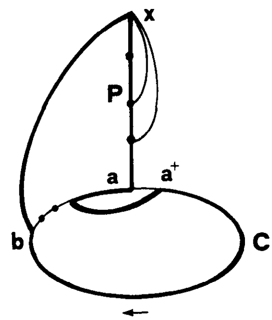

FIGURE 10

On the other hand, let v be any neighbour of x on C, different from a and b.

$$v \notin [b, a]_C$$

so $v^+ \notin \{b^{+l} | 1 \leqslant l \leqslant d_R(x)+1\}$ and a^+ is not adjacent to v^+ (Fig. 11). Last, a^+ is not adjacent to itself:

$$d_C(a^+) \leqslant |C| - 1 - (d_R(x)+1) - (d_C(x)-2)$$

$$d_C(a^+) \leqslant |C| - d_R(x) - d_C(x)$$

$$d_C(a^+) + d(x) \leqslant |C|.$$

Proof of the Theorem. Let a_1 be a vertex of C such that $d_R(a_1^+)$ is minimum. Let P be a path isuing from a_1, such that $V(P) \setminus \{a_1\} \subset R$ and of maximum length. Let x be the end of P in R and H the component of x in R.

G being k-connected, there exist k edges between H and C with distinct ends on C and we can suppose that one of these edges is the edge of P hav-

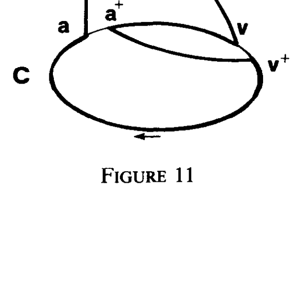

FIGURE 11

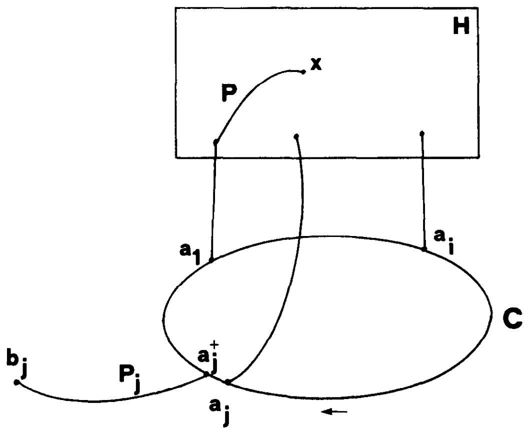

FIGURE 12

ing a_1 for an end (these edges have not necessarily distinct ends in H). We denote by $a_1, a_2, \ldots, a_k$ the ends of these edges on C. (See Fig. 12.) We set:

$$I = \{i \in \{1, 2, \ldots, k\} \mid d_R(a_i^+) = 0\}$$

$$J = \{j \in \{1, 2, \ldots, k\} \mid d_R(a_j^+) \neq 0\}.$$

If $j \in J$, let P_j be a path issuing from a_j^+ such that $V(P_j) \setminus \{a_j^+\} \subset R$ and of maximum length. Let b_j be the end of P_j in R.

We have the following properties:

* P_j does not intersect H (and P), since there is no C-path between a_j and a_j^+.

* If $j_1 \neq j_2$, P_{j_1} does not intersect P_{j_2}, since it is impossible to have a C-path between a_{j_1} and a_{j_2} (through H) and also a disjoint C-path between $a_{j_1}^+$ and $a_{j_2}^+$.

* The set $\{x\} \cup \{a_i^+ \mid i \in I\} \cup \{b_j \mid j \in J\}$ is an independent set.

* $d(x) \leqslant |C|/2$, by Lemma 3.

* For $j \in J$, $d(b_j) \leqslant |C|/2$, by Lemma 3.

First Case. $|I| \geqslant 2$. Let $p = |I|$, $I = \{i_1, i_2, \ldots, i_p\}$. We have: $d_R(a_{i_1}^+) = d_R(a_{i_2}^+) = \cdots = d_R(a_{i_p}^+) = 0$. And, from Lemma 4,

$$d_C(a_{i_1}^+) + d_C(a_{i_2}^+) \leqslant |C|$$

$$d_C(a_{i_2}^+) + d_C(a_{i_3}^+) \leqslant |C|$$

$$\vdots$$

$$d_C(a_{i_{p-1}}^+) + d_C(a_{i_p}^+) \leqslant |C|$$

$$d_C(a_{i_p}^+) + d_C(a_{i_1}^+) \leqslant |C|,$$

so that

$$\sum_{i\in I} d(a_i^+) \leqslant p(|C|/2).$$

Consequently,

$$m \leqslant d(x) + \sum_{i\in I} d(a_i^+) + \sum_{j\in J} d(b_j) \leqslant (k+1)\frac{|C|}{2}$$

$$|C| \geqslant \frac{2m}{k+1}.$$

Second Case. $|I|=1$. From the choice of a_1, $I=\{1\}$. From Lemma 5,

$$d_C(a_1^+) + d(x) \leqslant |C|.$$

Moreover, $d_R(a_1^+)=0$,

$$d(a_1^+) + d(x) \leqslant |C|$$

$$m \leqslant d(x) + d(a_1^+) + \sum_{j=2}^{k} d(b_j) \leqslant (k+1)\frac{|C|}{2}$$

$$|C| \geqslant \frac{2m}{k+1}.$$

Third Case. $|I|=0$.

$$m \leqslant d(x) + \sum_{j=1}^{k} d(b_j) \leqslant (k+1)\frac{|C|}{2}$$

$$|C| \geqslant \frac{2m}{k+1}.$$

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